MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2020

Calculator-assumed

Marking Key

The release date for this exam and marking scheme is 12th June.

Question 9 **(5 marks)**

Solution	
If P represents the point z and Q the point $2iz$ then the angle POQ is a right-angle If R denotes the point $(1+2i)z$ then OR is the diagonal of the quadrilateral; which is a rectangle The area of OPRQ is then the product of the lengths of the sides OP and OQ Since $z = r\,\text{cis}\,\theta$ the length of OP is just r As $OQ = 2iz$ the length of OQ is $2r$ Hence area of rectangle is $2r^2$	
Mathematical behaviours	**Marks**
• remarks that the angle POQ is 90 degrees • realises that the parallelogram is actually a rectangle • calculates the lengths of the sides of the rectangle • deduces the required area	1 1 1+1 1

Question 10 **(8 marks)**

Question 10(a) **(2 marks)**

Solution	
Now $\overrightarrow{AB} = -5\,\mathbf{i} - 10\,\mathbf{j}$ and $\overrightarrow{AC} = -5\,\mathbf{i} - 2\,\mathbf{k}$ lie in $\mathcal{P}$ Thus $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = -20\,\mathbf{i} + 10\,\mathbf{j} - 50\,\mathbf{k}$ is normal to $\mathcal{P}$	
Mathematical behaviours	**Marks**
• identifies two non-parallel vectors in $\mathcal{P}$ • calculates the cross product correctly	1 1

Question 10(b) **(2 marks)**

Solution	
Vector equation **r.n**=c for $\mathcal{P}$ is $-20x + 10y - 50z = -20 \times 5 = -100$ i.e. $2x - y + 5z = 10$	
Mathematical behaviours	**Marks**
• uses **r.n**=c • evaluates constant correctly	1 1

Question 10(c) **(3 marks)**

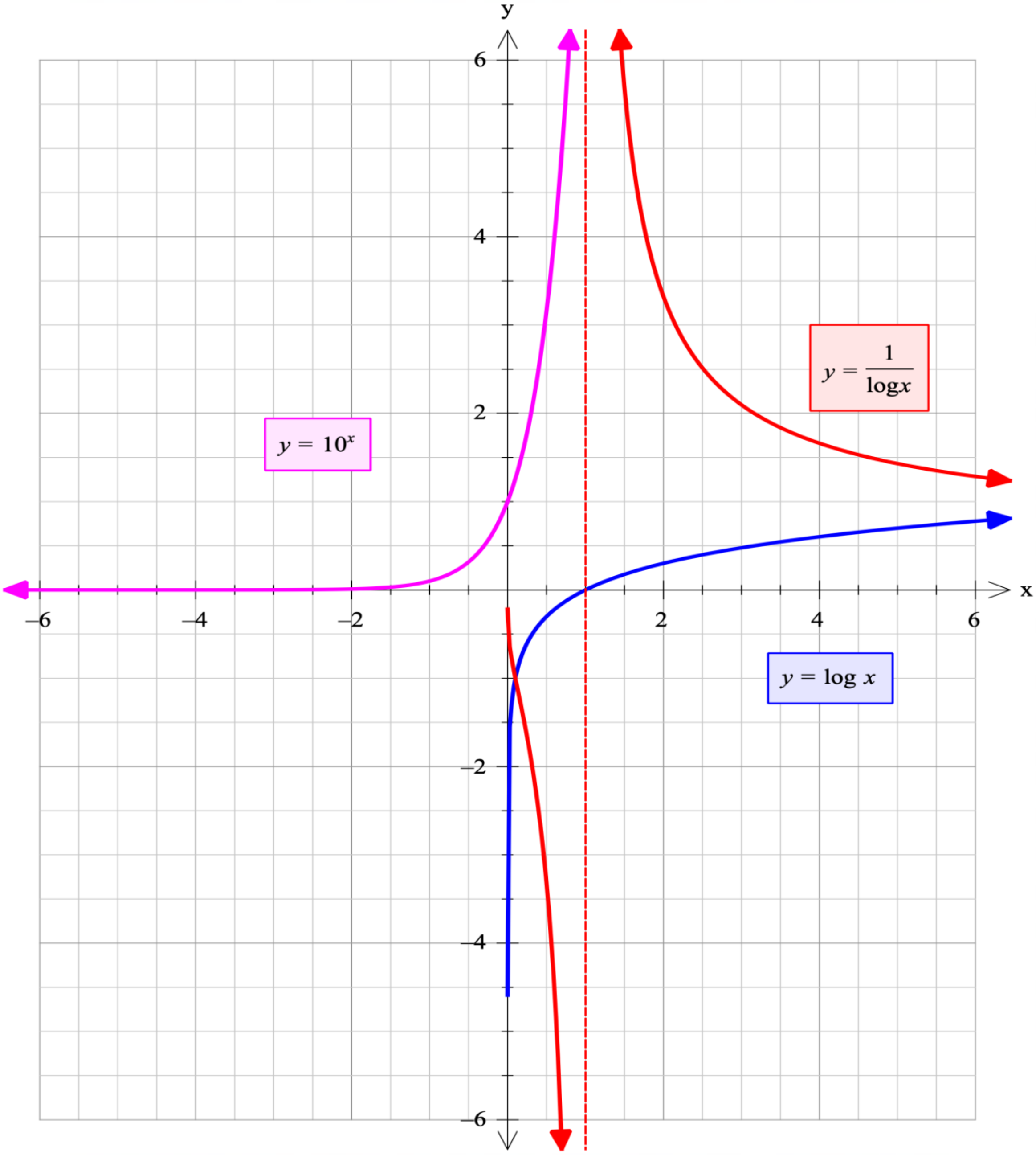
Solution	
Suppose that D has coordinates (a,b,c) Since $\overrightarrow{OD}$ is parallel to **n**, $(a,b,c)=t\,\mathbf{n}$ for some scalar t ie. $a=-20t,\ b=10t$ and $c=-50t$ (*) Since D lies on $\mathcal{P},\ 2a-b+5c=-300t=10$ (**) Hence $t=-\frac{1}{30}$ and the co-ordinates of D are $(a,b,c)=(-20t,10t,-50t)=\frac{1}{3}(2,-1,5)$	
Mathematical behaviours	**Marks**
• obtains equations (*) • obtains equation (**) • derives correct solution	1 1 1

Question 10(d) **(1 mark)**

Solution	
Distance OD= $\frac{1}{3}\sqrt{2^2+1^2+5^2}=\sqrt{10/3}\approx 1.83$	
Mathematical behaviours	**Marks**
• obtains correct result	1

Question 11 **(6 marks)**

Solution



Mathematical behaviours	Marks
• plots the correct shape of the graph of $y = 1/\ f(x)$ • demonstrates the correct behavior either side of the vertical asymptote	1 1+1
• displays the correct reflective property of $y = f^{-1}(x)$ in the line $y = x$ • shows the intersection point (0,1) exactly • shows that when $x = 1$ then y is approximately 3	1 1 1

Question 12 **(10 marks)**

Question 12(a) **(5 marks)**

Solution
Solving $\boldsymbol{r_A}(t) = \boldsymbol{r_B}(t')$ gives $(2+3t)\boldsymbol{i} + (t-15)\boldsymbol{j} + (11t-1)\boldsymbol{k} = 11\boldsymbol{i} + (3-3t')\boldsymbol{j} + (2+6t')\boldsymbol{k}$ So $2+3t = 11$, (*) $t-15 = 3-3t'$ (**) and $11t-1 = 2+6t'$ (***) So $t = 3$ and $t' = 5$ from (*) and (**) Note that (***) is satisfied when $t = 3$, and $t' = 5$, so the paths do intersect (at the point C$(11,-12,32)$.) The objects do not collide because they pass through the point of intersection of their paths at different times

Mathematical behaviours	Marks
• attempts to solve $\boldsymbol{r_A}(t) = \boldsymbol{r_B}(t')$	1
• obtains equations (*), (**) and (***)	1
• shows that the paths intersect	1
• states that the objects do not collide	1
• gives a valid reason	1

Question 12(b) **(5 marks)**

Solution
$\boldsymbol{r_A}(t) - \boldsymbol{r_B}(t)$ $= (2+3t)\boldsymbol{i} + (t-15)\boldsymbol{j} + (11t-1)\boldsymbol{k} - (11\boldsymbol{i} + (3-3t)\boldsymbol{j} + (2+6t)\boldsymbol{k})$ $= (3t-9)\boldsymbol{i} + (4t-12)\boldsymbol{j} + (5t-3)\boldsymbol{k}$ (*) So $\lVert\boldsymbol{r_A}(t) - \boldsymbol{r_B}(t)\rVert^2 = (3t-9)^2 + (4t-12)^2 + (5t-3)^2$ $= (9+16+25)t^2 - (54+96+30)t + (81+144+9)$ $= 50t^2 - 180t + 234$ (**) This is minimized when $t = \frac{180}{2\times 50} = 1.8$ When $t = 1.8$ $\boldsymbol{r_A} - \boldsymbol{r_B} = -3.6\boldsymbol{i} - 4.8\boldsymbol{j} + 6\boldsymbol{k}$; the minimum distance is $\sqrt{(-3.6)^2 + (-4.8)^2 + 6^2} \cong 8.49\ m$

Mathematical behaviours	Marks
• derives expression (*)	1
• obtains an expression for the square of distance between objects	1
• simplifies to derive (**)	1
• evaluates t correctly	1
• evaluates minimum distance correctly	1

Question 13 (9 marks)

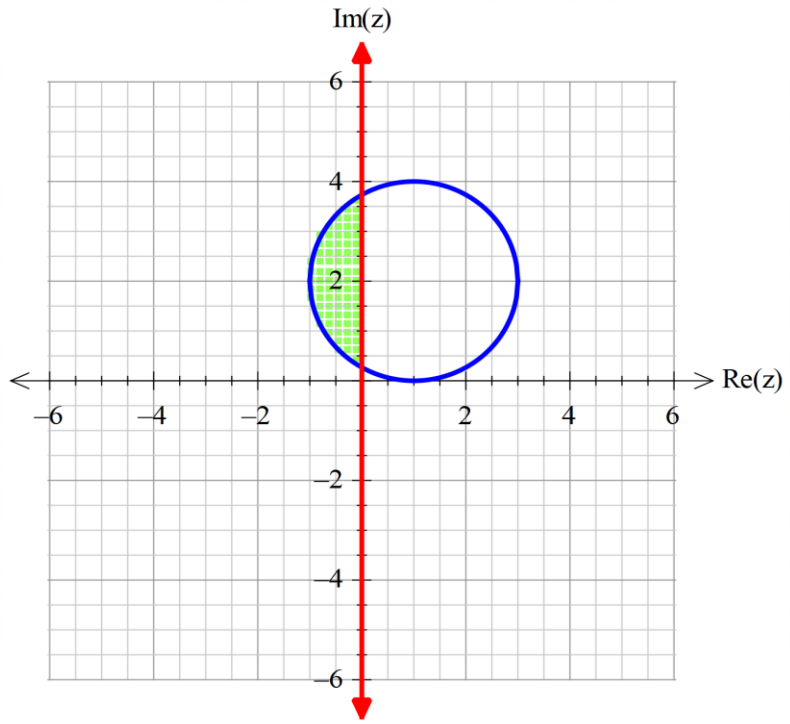
Question 13 (a) (5 marks)

Solution
If $\|z+1\|=\sqrt{2}\|z-i\|$ then with $z=x+iy$ have $\|(x+1)+iy\|=\sqrt{2}\|x+i(y-1)\|$
Hence $(x+1)^2+y^2=2\left[x^2+(y-1)^2\right]$
$\rightarrow\ x^2+y^2+2x+1=2x^2+2y^2-4y+2$
$\rightarrow x^2+y^2-2x-4y+1=(x-1)^2+(y-2)^2-4=0$
Hence $(x-1)^2+(y-2)^2=2^2$ so is a circle centre (1,2) and of radius 2

Mathematical behaviours	Marks
• expresses the equation in terms of x and y	1
• evaluates the modulus of both sides correctly	1
• simplifies the equation to the standard circle form	1
• determines the centre and radius of the circle	1+1

Question 13 (b) (4 marks)

Solution



As circle has centre (1,2) and has radius 2, it touches the x-axis. Cuts the y axis where

$$(y-2)^2=3 \Rightarrow y=2\pm\sqrt{3}$$

Mathematical behaviours	Marks
• shows circle in correct location with correct radius	1
• indicates circle touches the horizontal axis	1
• shades the correct area	1
• identifies the points where the circle cuts the vertical axis	1

Question 14 **(15 marks)**

Question 14(a) **(6 marks)**

Solution	
Since $f(x)=\dfrac{x^2+4x-10}{x-2}$, we can find the x-intercepts of $f(x)$ by setting the numerator to zero. Using the quadratic formula gives the roots of $f(x)=0$ as $x=-5.74$ and $x=1.74$ (2 dp) $$f'(x)=\frac{(x-2)(2x+4)-(x^2+4x-10)}{(x-2)^2}=\frac{x^2-4x+2}{(x-2)^2}.$$ The y-intercept is given by putting $x=0$ and $f(0)=5$ Solving $f'(x)=0$ gives $x=0.59$ and $x=3.41$. Point $x=2$ is also a critical point Since $f(0.59)=5.17$ and $f(3.41)=10.83$ then (0.59,5.17) and (3.41,10.83) are critical points.	
Mathematical behaviours	**Marks**
• determines the two roots of $f(x)=0$	1
• locates the y-intercept	1
• differentiates correctly using the quotient rule	1
• identifies $x=2$ as a critical point	1
• solves correctly $f'(x)=0$	1
• evaluates $f(0.59)$ and $f(3.41)$ for the two critical points	1

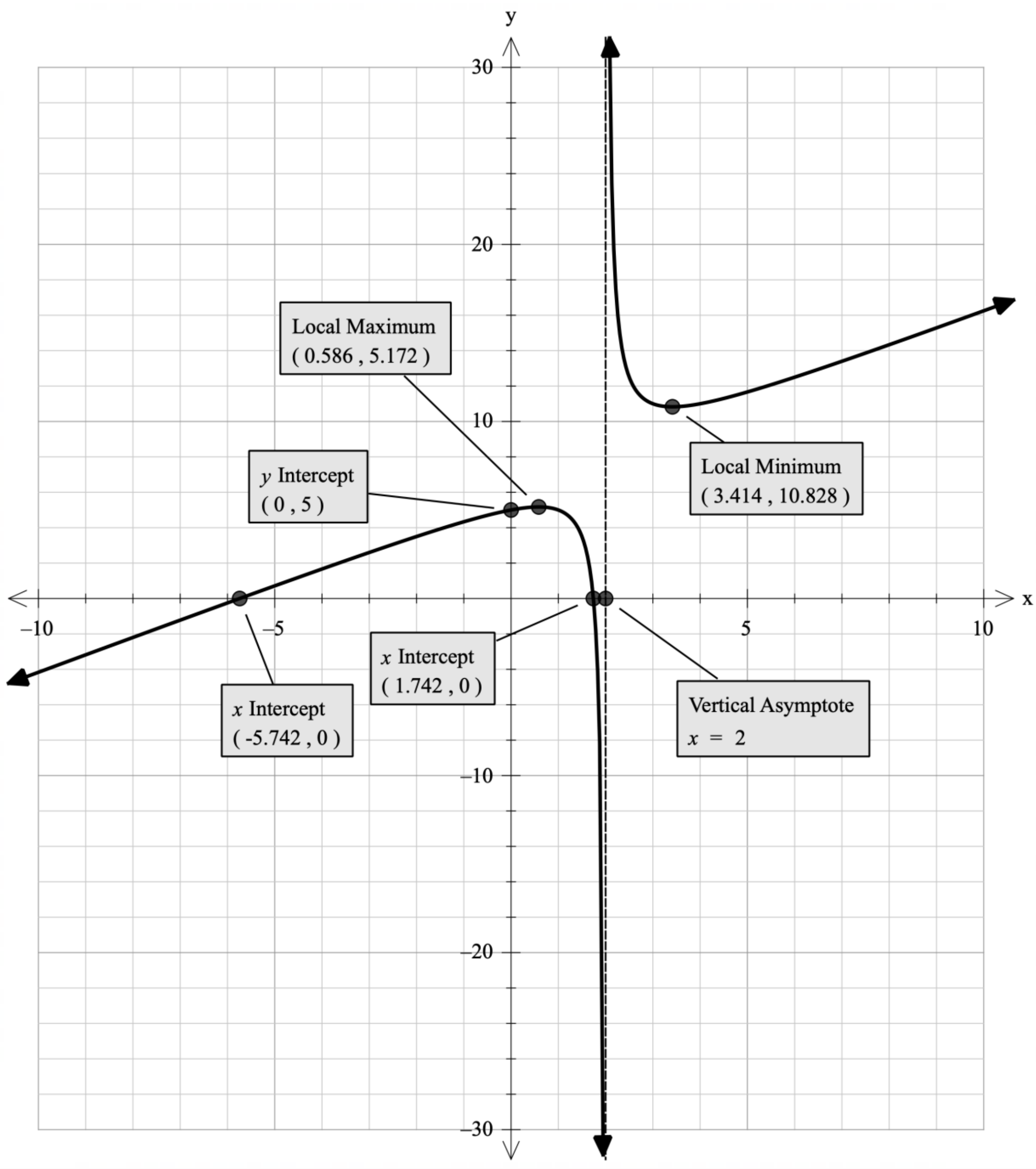
Question 14(b) **(3 marks)**

Solution	
Since $f(x)=\dfrac{(x-2)(x+6)+2}{(x-2)}=x+6+\dfrac{2}{x-2}$ it is clear that $f(x)\to x+6$ as $x\to\pm\infty$.	
Mathematical behaviours	**Marks**
• conducts the long division	1
• derives the form of the function for large $\|x\|$	1
• deduces the correct limiting behaviour as $x\to\pm\infty$	1

Question 14(c) **(1 mark)**

Solution	
The line $x=2$ is the vertical asymptote	
Mathematical behaviours	**Marks**
• identifies the vertical asymptote	1

Question 14(d) **(5 marks)**

Solution	
	

Mathematical behaviours	Marks
• identifies the vertical asymptote correctly • shows the critical points correctly • indicates where the curve cuts the y -axis • displays the inclined asymptote correctly	1 1+1 1 1

Question 15 **(8 marks)**

Solution
First we find an equation of the plane containing triangle ABC. Now $\overrightarrow{BC} = (4,0,0)$ and $\overrightarrow{BA} = (2,3,6)$ so that $\overrightarrow{BC} \times \overrightarrow{BA} = (4,0,0) \times (2,3,6) = (0,-24,12)$ Hence normal to the plane is parallel to $(0,2,-1)$ and equation of the plane is $2y - z = d$ for some d Substituting the coordinates of A (or B or C) gives $d = 7$. Since M lies in this plane $2y - z = 7$. (*) Now $\overrightarrow{BM} = (x-5, y-4, z-1)$ is perpendicular to $\overrightarrow{AC} = (2,-3,-6)$ Hence $2(x-5) - 3(y-4) - 6(z-1) = 0 \rightarrow 2x - 3y - 6z = -8$ (**) Also $\overrightarrow{AM} = (x-7, y-7, z-7)$ is perpendicular to $\overrightarrow{BC} = (4,0,0)$ so $$4(x-7) = 0 \Rightarrow x = 7$$ Now (**) becomes $-3y - 6z = -22 \Rightarrow y + 2z = \frac{22}{3}$ (***) Solving (*) and (***) gives $y + 4y - 14 = \frac{22}{3}$ or $y = \frac{64}{15}$. Then $z = 2y - 7 = \frac{23}{15}$. Hence co-ordinates of M are $\left(7, \frac{64}{15}, \frac{23}{15}\right)$.

Mathematical behaviours	Marks
• recognises that M lies in the plane containing triangle ABC	1
• obtains a normal to this plane	1
• derives equation (*)	1
• uses the orthogonality of the altitudes and the sides twice	1+1
• solves for x, y and z correctly	1+1+1

Question 16 (9 marks)

Question 16(a) (2 marks)

Solution	
We know that $z = cis(i\psi) = \cos\psi + i\sin\psi$ so that $cis(-i\psi) = \cos(-\psi) + i\sin(-\psi) = \cos\psi - i\sin\psi$ Then $z - z^{-1} = cis(i\psi) - cis(-i\psi) = \cos\psi + i\sin\psi - [\cos\psi - i\sin\psi] = 2i\sin\psi$	
Mathematical behaviours	**Marks**
• writes down appropriate form for $cis(-i\psi)$	1
• deduces the requisite result	1

Question 16(b) (3 marks)

Solution	
Now by de Moivre's theorem we have that $z^3 = \cos 3\psi + i\sin 3\psi$ Similarly $z^{-3} = \cos 3\psi - i\sin 3\psi$ so that $z^3 - z^{-3} = 2i\sin 3\psi$	
Mathematical behaviours	**Marks**
• uses de Moivre's theorem to get correct expression for z^3	1
• obtains corresponding result for z^{-3}	1
• deduces the stated result	1

Question 16(c) (4 marks)

Solution	
Now we have that $(z - z^{-1})^3 = z^3 - 3z + 3z^{-1} - z^{-3} = (z^3 - z^{-3}) - 3(z - z^{-1})$ Hence $(2i\sin\psi)^3 = (2i\sin 3\psi) - 3(2i\sin\psi)$ so $-8i\sin^3\psi = 2i\sin 3\psi - 6i\sin\psi \rightarrow -4\sin^3\psi = \sin 3\psi - 3\sin\psi$ or $\sin 3\psi = 3\sin\psi - 4\sin^3\psi$	
Mathematical behaviours	**Marks**
• expands the binomial expression correctly	1
• groups the terms on the right hand side to allow use of (a) and (b)	1
• substitutes the results of (a) and (b)	1
• rearranges to obtain the required result	1

Question 17 **(12 marks)**

Question 17(a) **(2 marks)**

Solution	
As $x(t) = 3\sin\alpha t$ and $y = 4\cos\alpha t$ then using $$\sin^2\alpha t + \cos^2\alpha t = 1 \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$ or, equivalently, $\frac{x^2}{9} + \frac{y^2}{16} = 1$	
Mathematical behaviours	**Marks**
• obtains the formulae for x and y	1
• eliminates the time correctly	1

Question 17(b) **(1 mark)**

Solution	
The path is an ellipse	
Mathematical behaviours	**Marks**
• states the shape correctly	1

Question 17(c) **(2 marks)**

Solution	
Velocity $\mathbf{v}(t) = 3\alpha\cos\alpha t\ \mathbf{i} - 4\alpha\sin\alpha t\ \mathbf{j}$ Acceleration $\mathbf{a}(t) = -3\alpha^2\sin\alpha t\ \mathbf{i} - 4\alpha^2\cos\alpha t\ \mathbf{j}$	
Mathematical behaviours	**Marks**
• states the correct expression for the velocity	1
• states the correct expression for the acceleration	1

Question 17(d) **(2 marks)**

Solution	
When $t = 0$ have $\mathbf{r}(0) = 4\,\mathbf{j}$ and the velocity $\mathbf{v} = 3\alpha\,\mathbf{i}$. Thus, at the most northerly end of the path (when y is greatest) then the velocity is in the positive **i** direction. Hence the car moves in a clockwise direction around the path.	
Mathematical behaviours	**Marks**
• draws the correct conclusion	1
• gives a valid reason for drawing the conclusion	1

Question 17(e) **(2 marks)**

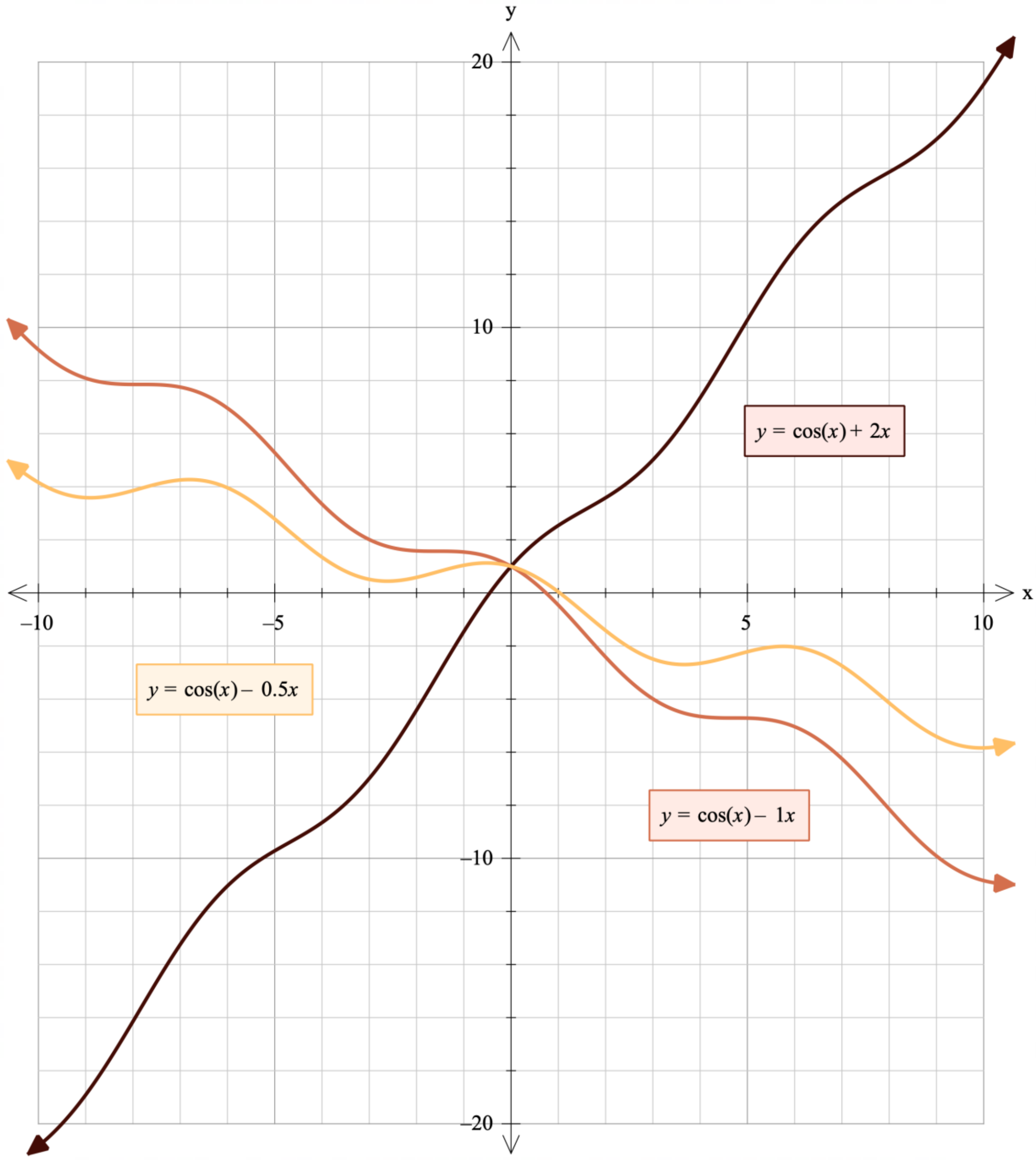
Solution	
If the car completes a circuit in 74 seconds then $$74\alpha = 2\pi \Rightarrow \alpha = 0.0849$$ correct to 3 s.f.	
Mathematical behaviours	Marks
• uses $74\alpha = 2\pi$ • derives the correct value to the required accuracy	1 1

Question 17(f) **(3 marks)**

Solution	
Since $\mathbf{v}(t) = 3\alpha\cos\alpha t\ \mathbf{i} - 4\alpha\sin\alpha t\ \mathbf{j}$ then $$v^2 = 9\alpha^2\cos^2\alpha t + 16\alpha^2\sin^2\alpha t = 9\alpha^2 + 7\alpha^2\sin^2\alpha t$$ since $\cos^2\alpha t + \sin^2\alpha t = 1$. Hence maximum value of v^2 occurs when $\sin^2\alpha t = 1$ so then $$v^2 = 16\alpha^2 \Rightarrow v = v_{\max} = 4\alpha = 0.340$$ Hence max speed is 34 cm per second approximately.	
Mathematical behaviours	Marks
• recognizes speed as the length of the velocity vector • derives expression for the square of the speed • obtains the correct answer	1 1 1

Question 18 **(9 marks)**

Question 18(a) **(3 marks)**

Solution	
	
Mathematical behaviours	**Marks**
• plots the three graphs reasonably accurately	1+1+1

Question 18(b) **(3 marks)**

Solution	
Now $f'(x) = A - \sin x$ so that for values x we have that $A-1 \le f'(x) \le A+1$ The function is one-to-one if its derivative never changes sign Hence the function is one-to-one if either $A \le -1$ or $A \ge 1$	
Mathematical behaviours	Marks
• notes the range of values taken by the derivative • states correctly the criterion for a 1-1 function • obtains the correct ranges for A	1 1 1

Question 18(c) **(3 marks)**

Solution	
To evaluate $f^{-1}(4)$ we need to solve the equation $4 = \cos x + 3x$. This has the solution $x \simeq 1.22$	
Mathematical behaviours	Marks
• writes down the equation to be solved • uses calculator to determine root • quotes answer to specified accuracy	1 1 1

Question 19 (9 marks)

Question 19(a) **(3 marks)**

Solution	
If $$P(z) = z^4 + 4z^3 + 9z^2 + 16z + 20$$ Then $P(2i) = (2i)^4 + 4(2i)^3 + 9(2i)^2 + 16(2i) + 20 = 16 - 32i - 36 + 32i + 20 = 0$ Hence one root of $P(z) = 0$ is $z = 2i$ as required	
Mathematical behaviours	Marks
• substitutes $z = 2i$ in equation for $P(z)$ • simplifies real parts and imaginary parts to show that $P(2i) = 0$	1 1+1

Question 19(b) **(6 marks)**

Solution
As $P(z)$ has real-valued coefficients, the roots of $P(z)=0$ come in complex conjugate pairs. Then since $z=2i$ is a solution then so is $z=-2i$ and hence z^2+4 is a factor of $P(z)$ By long division we have that $P(z)=z^4+4z^3+9z^2+16z+20=(z^2+4)(z^2+4z+5)$ Now $z^2+4z+5=0$ has complex-valued roots $z=\dfrac{-4\pm\sqrt{16-20}}{2}=-2\pm i$ Hence the four roots of $P(z)=0$ are $z=\pm 2i$ and $z=-2\pm i$

Mathematical behaviours	Marks
• states that roots of the equation come in complex conjugate pairs	1
• writes down the solution $z=-2i$	1
• deduces a quadratic factor of $P(z)$	1
• performs the long division to find the other quadratic factor	1
• solves the quadratic for the other two roots	1
• states the four roots of $P(z)=0$	1